Applied Mathematics and Mechanics (English Edition) Vol. 26 No. 7 Jul. 2005
Published by Shanghai University, Shanghai, China
©Editorial Committee of Appl. Math. Mech., ISSN 0253-4827
Article ID: 0253-4827(2005)07-0929-09

CURVATURE RELATION OF WAVE FRONT AND WAVE CHANGING IN EXTERNAL FIELD *

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(Communicated by DAI Shi-qiang)

Abstract: The changing of wave structure in excitable media in external field is studied and the curvature relation of wave front is analyzed. Under external stimulus the normal velocity of wave front has linear relation with mean curvature of wave front, plane velocity and external field. The simulation methods have been used to analyze Bar-Eiswirth model with external field and obtain the wave pattern of excitable media contained external stimulus. These theoretical analysis and simulation results are identical with experiments of BZ reaction. So the results here theoretically explain the BZ phenomenon under external field and the simulation results here have rich wave patterns.

Key words: excitable medium; spiral wave; pattern; external field
Chinese Library Classification: O534; O362 Document code: A
2000 Mathematics Subject Classification: 92C45; 74J30

Introduction

The studies of excitable media concern with BZ reaction, slime mold aggregation and cardiac tissue[1]. The properties of these media are: for small perturbations the media quickly recover to their rest states, while for stimuli exceeding a threshold the media will be activated and remain in excited states for a long period. In excitable media research, it is commonly accepted that cardiac fibrillation is caused to excitable nature of cardiac tissue[2]. An opinion is that action potential of cardiac cell is able to form spiral patterns and breakups which will affect the rhythm of heart and evoke cardiac fibrillation. If cardiac fibrillation is not treated in time, a person will die in several minutes. As it is concerned with human life, direct experiments on cardiac tissue are difficult. Instead, BZ reaction serves as an analogue for the understanding of cardiac tissue, since both BZ reaction reagents and cardiac tissue are excitable while BZ reaction is much tractable for experiments.

* Received Apr. 22, 2003; Revised Apr. 16, 2005
Project supported by the National Natural Science Foundation of China (No. 19902005)
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The known wave patterns of BZ experiments and slime mold aggregation include spirals, double spirals, super-spirals and multi-spirals in the planar media[2]. In some situation, the tip of the spiral meanders and the loci of these meandering reveal twisted spiral chain or petal shape[3]. Patterns formed in spatial media are more plentiful, which includes ring scroll wave, twisted scroll and stair scroll waves. However, it is difficult for observation due to the limitation of planar representation of spatial patterns. Furthermore, the interaction of the spatial waves makes the description of these patterns difficult as well. Simulation results show that there are spirals and their breakups in planar media and scroll waves with the twisted, knotted and linked organization centers[2] in spatial media. For BZ reaction the dynamic model can be simplified as the following equations[4]:

$$\frac{\partial u}{\partial t} = \varepsilon \nabla^2 u + \varepsilon^{-1} f(u, v), \quad \frac{\partial v}{\partial t} = \varepsilon \mu \nabla^2 v + g(u, v), \tag{1}$$

where u denotes fast variable and v denotes show variable. ε is the small parameter and μ denotes the ratio of the diffusion coefficient. Different reaction terms $f(u, v)$, $g(u, v)$ express different dynamic model, but they all possess common properties of excitable media: the nulline of f where $f(u, v) = 0$ have three branches $u = u_-(v)$, $u = u_0(v)$ and $u = u_+(v)$ and the nulline of g is where $g(u, v) = 0$ changes monotonously. The intersection of these two lines locates on $u = u_-(v)$, the left branch of $f(u, v) = 0$. The excited state of the media exists in certain region of the space where it has two surfaces as the boundaries. We use fast variable u to describe the situation when the boundary surface π jumps from rest state $u = u_-(v)$ to excited state $u = u_+(v)$ which is called wave front. The situation when the boundary surface recovers from excited state $u = u_+(v)$ to rest state $u = u_-(v)$ is called wave back. The intersection of wave front and wave back can be abstracted as a spatial curve which is called organization center. u changes slowly near the tangent plane of the wave front and changes rapidly in the normal direction while the change of the slow variable v is not as obvious.

BZ reaction can produce rich wave patterns, which are affected by many factors including external stimulus, periodic impelling, media uniformity and boundary. The known BZ reaction shows that when electric field is applied to the BZ reagent, the wave will slowly move toward the anode[5]. In the direction perpendicular to the electrical field, the spiral will evolve apart. If a periodic electric field is applied, there appear the structures of super-spirals with big spiral and many small spiral waves[6]. When intensive lights are applied, the small waves or breakup spiral waves produced by the reagents form labyrinthian standing patterns. If the frequency of the external light are changed, resonance with spiral wave may result[6,7]. Temperature field also affects the wave structure. When temperature changes, various twisted scroll waves and twisted organization centers are formed. In temperature field the plane generated by organization center of scroll wave will move slowly and finally become perpendicular to the temperature gradient field[8]. Besides external field, the shape of the boundaries, anisotropic nature of the media and others are also considered as sources of external stimulus.

Most of the recent BZ experiments study the influence of external stimulus to the BZ reactions. However, few of them include theoretic analysis and numerical simulation results. Here in this paper, perturbation method is used to obtain theoretically the wave front curvature relation of excitable media in external field. Then simulations on the Bar-Eiswirth excitable model[9,10] are carried out and the observed wave changes in external field are showed.

2 Perturbation of Model

The influence of external field to excitable media is complicated. Different theoretic models describe the influence in different ways. The effects of external field to BZ reaction is by modifying the rate of change of reagent concentration and can be expressed by the change of reaction term. For fixed external field, it is described by the following Steinbock model[7]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \nabla^2 u + \varepsilon^{-1} f(u, v) + M_1 \dfrac{\partial u}{\partial x} + M_2 \dfrac{\partial u}{\partial y} + M_3 \dfrac{\partial u}{\partial z}, \\ \dfrac{\partial v}{\partial t} = \varepsilon\mu \nabla^2 v + g(u, v), \end{cases} \tag{2}$$

where the vector $\boldsymbol{M} = \{M_1, M_2, M_3\}$ denotes the effects of external field to the whole model. For simplicity, we only consider case of two dimensions. With the method in Ref. [3], one can establish moving coordinate system in the boundaries of wave fronts. Let curve Γ denote wave front, s denotes the arc length from wave tip to the point of wave front Γ, and r denotes the normal distance from the point (x, y) in the neighbor wave front to Γ. A local orthogonal moving coordinate system (r, s) in the neighborhood of wave front is established. For any moment t, the point (x, y) in the neighborhood of wave front can be described in local coordinate system (r, s) in the boundary and the wave front set Γ is denoted by point set $\{(x, y): r(x, y, t) = 0\}$. Without loss of generality, we choose $|\nabla r| = 1$ and $\nabla r \cdot \nabla s = 0$. Suppose the transformation between moving coordinate and original coordinate is

$$\tau = t, \qquad r = r(x, y, t), \qquad s = s(x, y, t).$$

The transformation relations of these two coordinate systems are

$$\begin{cases} \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial \tau} + r_t \dfrac{\partial}{\partial r} + s_t \dfrac{\partial}{\partial s}, \qquad \dfrac{\partial}{\partial x} = r_x \dfrac{\partial}{\partial r} + s_x \dfrac{\partial}{\partial s}, \\ \dfrac{\partial^2}{\partial x^2} = r_x^2 \dfrac{\partial^2}{\partial r^2} + s_x^2 \dfrac{\partial^2}{\partial s^2} + r_{xx} \dfrac{\partial}{\partial r} + s_{xx} \dfrac{\partial}{\partial s} + 2 r_x s_x \dfrac{\partial^2}{\partial r \partial s}, \\ \nabla^2 = \dfrac{\partial^2}{\partial r^2} + |\nabla s|^2 \dfrac{\partial^2}{\partial s^2} + \nabla^2 r \dfrac{\partial}{\partial r} + \nabla^2 s \dfrac{\partial}{\partial s}. \end{cases}$$

In moving coordinate system, Eq. (2) is changed into

$$\begin{cases} \varepsilon\left(\dfrac{\partial u}{\partial \tau} + \dfrac{\partial u}{\partial r} r_t + \dfrac{\partial u}{\partial s} s_t\right) = \varepsilon^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\partial^2 u}{\partial s^2}|\nabla s|^2 + \dfrac{\partial u}{\partial r}\nabla^2 r + \dfrac{\partial u}{\partial s}\nabla^2 s\right) \\ \qquad + \varepsilon M_1\left(r_x \dfrac{\partial u}{\partial r} + s_x \dfrac{\partial u}{\partial s}\right) + \varepsilon M_2\left(r_y \dfrac{\partial u}{\partial r} + s_y \dfrac{\partial u}{\partial s}\right) + f(u, v), \\ \dfrac{\partial v}{\partial \tau} + \dfrac{\partial v}{\partial r} r_t + \dfrac{\partial v}{\partial s} s_t = \varepsilon\mu\left(\dfrac{\partial^2 v}{\partial r^2} + \dfrac{\partial^2 v}{\partial s^2}|\nabla s|^2 + \dfrac{\partial v}{\partial r}\nabla^2 r + \dfrac{\partial v}{\partial s}\nabla^2 s\right) + g(u, v). \end{cases} \tag{3}$$

In the neighborhood of wave front, the fast variable u changes too rapidly in the normal direction for observation. For easier observation, the unit along normal direction is scaled up as $r = \varepsilon R$. Hence the approximation of Eq. (3) in order $O(\varepsilon^0)$ is

$$\begin{cases} \dfrac{\partial^2 u}{\partial R^2} + (\varepsilon \nabla^2 r + M_1 r_x + M_2 r_y - r_t) \dfrac{\partial u}{\partial R} + f(u, v) = 0, \\ \dfrac{\partial v}{\partial R} r_t = \mu \dfrac{\partial^2 v}{\partial R^2}, \end{cases} \tag{4}$$

where $\nabla^2 r$ indicates the mean curvature of wave front surface. In order to analyze the effects of bending, the term $\varepsilon \nabla^2 r$ is preserved. In the second term of Eq. (4), $v = v_0$ is set to a constant in the neighborhood. Equation (4) can be further simplified as follows:

$$\frac{\partial^2 u}{\partial R^2} + (\varepsilon \nabla^2 r + M_1 r_x + M_2 r_y - r_t) \frac{\partial u}{\partial R} + f(u, v_0) = 0 \tag{5}$$

For Eq. (5), the domain of variable R is $(-\infty, +\infty)$. On both sides of wave front in the normal direction variable u jump greatly from $u_-(v_0)$ to $u_+(v_0)$. With the property of one-dimensional reaction diffusion equation we know

$$\lim_{R \to -\infty} \frac{\partial u}{\partial R} = 0, \qquad \lim_{R \to +\infty} \frac{\partial u}{\partial R} = 0.$$

Multiplying $\dfrac{\partial u}{\partial R}$ on both sides of Eq. (5) and integral the variable R from $-\infty$ to $+\infty$. Pay attention the limits condition we obtain

$$r_t = \varepsilon \nabla^2 r + M_1 r_x + M_2 r_y + c, \tag{6}$$

where $c = \dfrac{\int_{u_-(v_0)}^{u_+(v_0)} f(u, v_0)\,\mathrm{d}u}{\int_{-\infty}^{+\infty} \left(\dfrac{\partial u}{\partial R}\right)^2 \mathrm{d}R}$ denotes the velocity of the plane wave and $r(t)$ denotes the normal distance from point (x, y) in the neighborhood of wave front to the wave front surface. In Eq. (6) r_t denotes the normal velocity of wave fronts, $\nabla^2 r$ expresses the effect of mean curvature of wave front, $M_1 r_x$, $M_2 r_y$ denote the influence of external field to normal velocity. The relation (6) is the linear equation of normal velocity of wave front to the mean curvature of wave front and external field. Let vector $\boldsymbol{M} = \{M_1, M_2\}$, θ denote the angle between normal wave front and external field $\boldsymbol{M}$ and K denote the mean curvature of wave front. Equation (6) can be rewritten as follows:

$$N = \varepsilon K + M \cos\theta + c, \tag{7}$$

where $M = |\boldsymbol{M}| = \sqrt{M_1^2 + M_2^2}$.

Similar to the planar case, we can obtain the spatial curvature relation. It expresses the linear relation between normal velocity of wave front and the stimulus of external field. From Eq. (7), the following conclusion can be obtained. Under the influence of external field, the normal velocity of wave front is affected by two factors. One is mean curvature of wave front and the other is external field. They explain the known BZ experiments results well.

For spiral wave, the magnitude of velocity is affected by the absolute value of external field, the angle between external field and the wave front. When $\theta = 0$, the normal direction of wave front is the same as external field, while the normal velocity of wave front reaches its maximum. When $\theta = \pi$, normal direction is inverse to the external field while the normal velocity is at its minimum. In external electrical field, the spiral reaches its maximum velocity in the direction of external field and minimum in the opposite direction. As the normal velocity reaches its maximum in electrical direction, the whole wave will move toward this direction. This is consistent with BZ experiments[5].

For scroll wave, waves with ring organization center are considered instead. When the scroll wave is influenced by the external field, the coordinal velocity of wave front will reach is maximum. The whole scroll wave will move toward the direction of external field. During the moving, it can be seen from Eq. (7) that the moving speed on the wave is different. For wave in the center of the organization, only when external field is perpendicular to the plane containing the center, velocities in different directions can reach their balance. That is stable states of the scroll wave. The results here explain the BZ experiments in temperature field[8]. For spiral breakup, in the direction of external field, the wave velocity will increase and the results here explain the moving phenomena of breakup wave as well.

3 Numerical Results of External Field

The changing of the waves pattern generated by BZ experiments is affected by many factors including external field, anisotropic nature of media and the shape of boundaries. Since the model is nonlinear, theoretic analysis is difficult. Some researchers used sectional functions to model the anisotropic nature of the media and the properties of boundary. Here simulation is used to analyze the effects of external field on Bar-Eiswirth model[8,10]. The diffusion term is approximated with Euler implicit grid method and the reaction diffusion simulates the operator splitting method[10]. Rectangular boundary condition is applied.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \nabla^2 u + \varepsilon^{-1} u(1-u)\left(u - \dfrac{v+b}{a}\right) + M_1 \dfrac{\partial u}{\partial x} + M_2 \dfrac{\partial u}{\partial y}, \\ \dfrac{\partial v}{\partial t} = \varepsilon \mu \nabla^2 v + u^3 - v. \end{cases} \tag{8}$$

The simulation of the model (8) carried out on EZSPIRAL which runs in Linux system if the model is not affected by external field. Here the modified model runs in windows system and external stimuli are also considered. Because of nonlinear nature of the problem, many factors may affect the simulation results. The simulation uses 5 points difference method to approximate the diffusion term and the area is fixed in a square of 80×80. Other parameters are $a = 0.75$, $b = 0.08$, $\varepsilon = 1/80$, $\delta = 1E-4$, $\mu = 0$, respectively. For each direction we choose 242 points with a simulation time step of 0.8. The simulation result is showed below according to variable v.

In Fig. 1, it can be seen that the system forms spiral wave without disturbance and the spiral wave will move and finally disappear with application of external stimulus. These numerical results are consistent with both the theoretical analysis in Section 2 and BZ experiments in literatures. However the numerical results with external stimulus are more vivid than the theoretical analysis. If parameter is set to $\varepsilon = 1/20$ and other parameters remain unchanged, the undisturbed patterns would show spiral structure just as Fig. 1. The disturbed patterns reveal spiral breakup and these breakup waves contain rich patterns during long period of simulation as illustrated in Fig. 2.

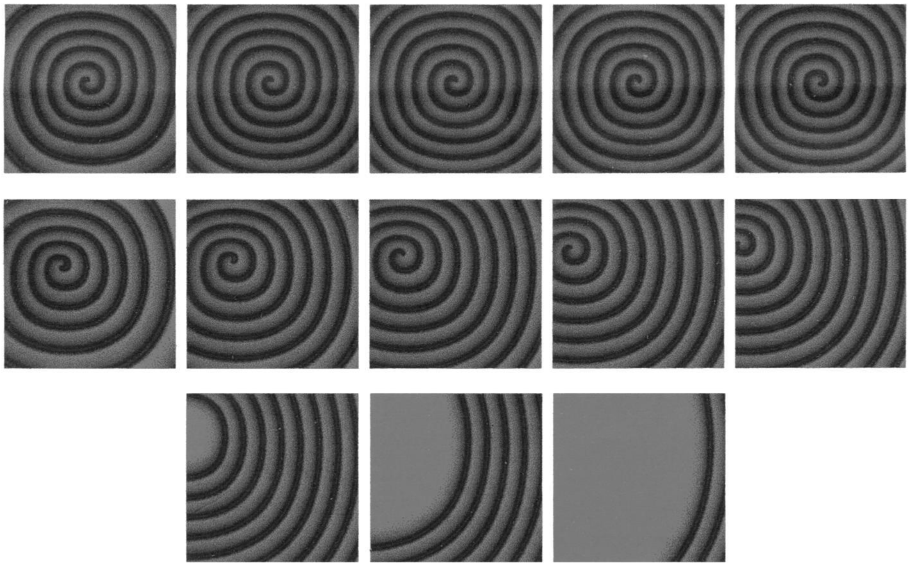

Fig 1 The pictures in the first line correspond to the undisturbed wave. The pictures in the second line correspond to the disturbed pictures with coefficients $M_1 = 1, M_2 = 0$. The moment of sampling for the simulation are $t = 20, 30, 40, 50, 60$ s respectively. The third line shows the gradual disappearance of the wave after disturbance

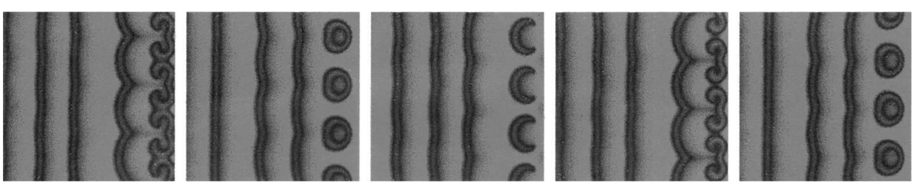

Fig 2 The simulation time are $t = 1\,713, 1\,720, 1\,727, 1\,730, 1\,737$ s. After long period of simulation, the structures become stable

Many factors can affect the wave patterns of excitable media. If the parameters in Fig 2 remain unchanged while the number of the points in each direction is set to 121, the resulting patterns are different as illustrated in Fig 3.

It can be seen that changing the number of points in each direction will result in changing of the wave patterns. This is mainly due to the effects of anisotropic nature of the media or boundary conditon. When the number of points in each direction is denser, the effects of anisotropic property or boundary condition become less observed. On the contrary, when the number of points is sparse, the effects can be easily observed and various breakup wave patterns would form. This phenomenon is also observable in many known BZ experiments. However, the theoretic analysis of this phenomenon is rather difficult.

In Figs 1–3, the diffusion term is calculated with 5-point Laplacian. However, with 9-point Laplacian, another kind of spiral waves results. The original spiral wave will disappear after a certain period of simulation. Figure 4 is obtained by setting the square to 80×80, the number of points to 242, time step to 0.8 and other parameters to $a = 0.75$, $b = 0.06$, $\varepsilon =$

$1/14$ $\delta = 1E-4$ $\mu = 0$

In Fig 4 the undisturbed waves are spiral waves and their tips show meandering When the external stimulus appears the waves first break up and then disappear Comparing Fig 1 and Fig 4 it seems that there is a type of spiral wave that will disappear under the influence of external field The conditions of such disappearance need further studies One possible condition may be that if the wave keeps evolving for a long period under external stimulus it must be breakup wave.

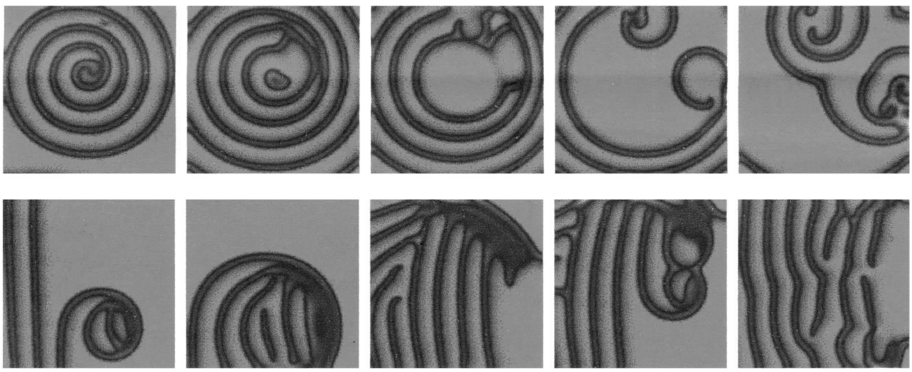

Fig 3 The pictures in first line correspond to undisturbed wave pattern and the moment of sampling are $t = 30$ 40 50 60 70 s respectively. The second line are disturbed pictures with coefficients $M_1 = 1$, $M_2 = 0$ and they are sampled at $t = 100$ 110 130 140 200 s respectively

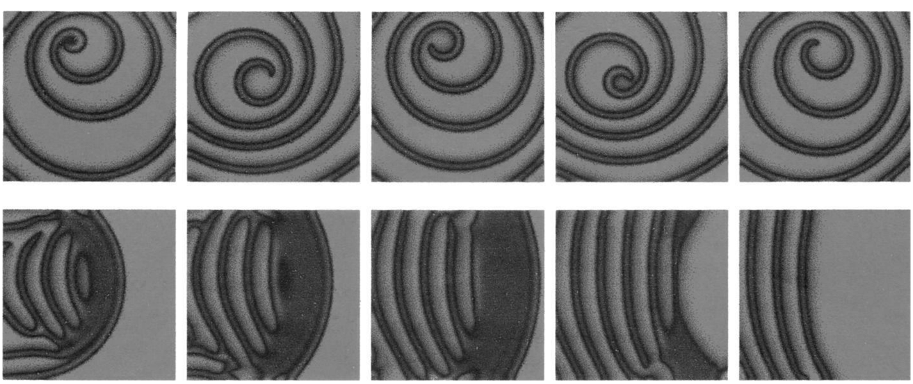

Fig 4 Pictures in the first line show undisturbed spiral waves and the sampling moments are $t = 80$ 120 150 1 221 1 242 s respectively Pictures in the second line are disturbed waves with coefficients $M_1 = 1$, $M_2 = 0$ The breakup wave gradually disappears as is showed in moment $t = 30$ 40 50 60 70 s

If 5-point Laplacian is adopted in a square of 80×80 with 242 points in each direction other parameters are $a = 0$ 75 $b = 0$ 01 $\varepsilon = 1/12$ $\delta = 1E-4$ $\mu = 0$ and time step is 0.8 the wave will soon disappear when there is no external stimulus or spiral breakup will appear The result is shown in Fig 5.

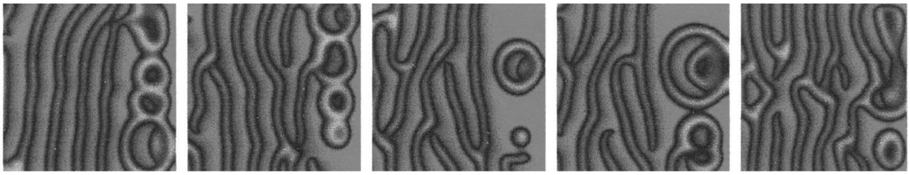

Fig 5 The sampling moments are $t = 152$, 446, 511, 514, 1 314 s with coefficients are $M_1 = 1$, $M_2 = 0$. The wave patterns breakup wave which moves for long period, but the wave will disappear when the external stimulus is absent

From the above numerical results, it can be seen that the external stimulus will obviously influence the wave pattern. As the changing of normal velocity, the most obviously effects of external stimulus are the global moving of whole spiral pattern. In external field and with the influence of other factors there occurs the shape transformation between spiral wave and spiral wave breakup and the disappearance of the spiral wave. When spiral wave appears in external stimulus, it will disappear eventually as the moving of spiral wave. Only the breakup wave pattern can remain for a long period. In the sense of pattern preservation, breakup wave is stable. However the change of spiral wave is not the result of a single factor and a thorough analysis of change of the wave pattern is still difficult

4 Conclusions

In this paper, we analyze the influence of external field to the wave structure of excitable media. The theoretical analysis describes the curvature relation of wave front surface in excitable media. The normal velocity of wave front has linear relation with mean curvature, plane velocity and external field. This relation reveals that the normal velocity of wave front will increase in the direction of external field. It gives an explanation to the BZ experiments resulting under temperature field or electric field. The simulation results here show rich spiral wave patterns. In external field, one can see the movement of the whole wave pattern as well. This is consistent with theoretic analysis and BZ experiments phenomena. Simulation results indicate that the spiral wave can move in external field when there is no external stimulus. The moving spiral can disappear immediately or remain breakup for a long period. The breakup results reveal the interaction of multi-spiral, stripe stand wave, water labyrinthian wave and island connection wave pattern[1,2]. For spiral breakup or wave patterns which disappear rapidly, rich patterns result in external stimulus as well.

By examining the change of wave, one can find that wave pattern transformation in excitable media is not only the effect of external field but also the result of interaction of many factors. In particular, the boundary shape and the anisotropic nature of the media is also the source for new wave patterns. Due to the complexity of theoretic model, the analysis in this paper did not cover external periodic stimulus, anisotropic media and boundary shape other than rectangle. Rather they are left as furture studies

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