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STRUCTURE OF WAVE FRONT AND ORGANIZATION CENTER IN EXCITABLE MEDIA *

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Abstract: *With help of establishing the moving coordinate on the wave front surface and the perturbation analysis in the boundary layer, the structures of wave front and organization center in excitable media were studied. The eikonal equation of wave front surface and general equation of organization center were obtained. These eikonal equations reveal the wave front surfaces have structures of twisted scroll wave and Möbius band, the organization centers have structures of knotted and linked ring. These theoretical results not only explain the wave patterns of BZ (Belousov-Zhabotinskii) chemical reaction but also give several possibility structures of wave front surface and organization center in general excitable media.*

Key words: excitable media; wave front; organization center

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Introduction

BZ reaction has rich wave patterns[1], When BZ reagents place on a thin dish, experiment results reveal that the reagents appear the pattern of spiral, double spiral and super spiral[2]. Under certain circumstance the tip of spiral will meander and the locus of meander appear in spiral chain or twisted flower. When BZ reagents put in a tube the wave structures change complicated. The waves front reveals twisted surface, scroll wave and stair shape. The organization centers reveal in ring, spiral and its twisted etc[3,4]. Many phenomena have the same mechanism with BZ reaction[5] and in the same wave patterns, such as the aggregation of social amoebae and cardiac tissue etc[6,7]. These phenomena and BZ reaction are excitable media and can be theoretically abstract in the following model:

$$\varepsilon \frac{\partial u}{\partial t} = \varepsilon^2 \nabla^2 u + f(u, v), \qquad \frac{\partial v}{\partial t} = \varepsilon\tau \nabla^2 v + g(u, v). \tag{1}$$

Here u is called fast variable and v is slow variable. Distinct functions $f(u, v)$, $g(u, v)$ express

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different model[8]. The common properties of Eq. (1) are: The null line of $f(u,v)=0$ has three branches $u=u_-(v)$, $u=u_0(v)$ and $u=u_+(v)$. The null line of $g(u,v)=0$ changes monotonously. The intersection of these two null lines locates on the left branch $u=u_-(v)$ of $f(u,v)=0$. If we add Neumann boundary condition in Eq. (1) the numerical results show the plane patterns appear the spiral structure, the locus of tip appears in twisted chain or twisted ring. The three dimension patterns reveal scroll structure and twisted, the organization centers have spiral shape in ring and their twisted, knotted and linked. For linked and knotted organization center it is difficult to understand the relation between wave surface and its organization centers[9].

The theoretic analysis of Eq. (1) reveals the normal velocity of wave surface and mean curvature have linear relation[10,11]. As complicated in form this relation can explain simple spiral patterns but helpless for normal structure. In this paper the perturbation method has been employed to obtain the characteristic equation of wave fronts and organization centers in excitable media and to analyze the structure contained in characteristic equation.

1 Perturbation Analysis

The excited states of excitable media exist in a region of space. This region contains two surfaces as its boundary. We use fast variable u to describe it: The surface jump from rest state $u=u_-(v)$ to excited state $u=u_+(v)$ is called wave front π. The surface recovery from excited to rest state is called wave back. The intersection of wave front and wave back can be abstracted as a line, called organization center. In the neighborhood of tangent direction of wave front the variable u changes little and in the normal direction u changes fast. The variable v is almost unchanged.

For any time t, we introduce a local moving orthogonal coordinate system (s,p) on wave front π, if r denoted the normal distance from a point in the neighborhood of π to wave front π. In the neighborhood of wave fronts π, there is a new moving coordinate system (r,s,p). Obviously, the set $\{(x,y,z): r(x,y,z,t)=0\}$ denoted the wave front π.

Suppose the transformations from original coordinate system (x,y,z) to new moving coordinate system (r,s,p) are follows:

$$\tau=t,\quad r=r(x,y,z,t),\quad s=s(x,y,z,t),\quad p=p(x,y,z,t). \tag{2}$$

Without losting generalization, we set $|\nabla r|=1$, $\nabla r\circ\nabla s=0$, $\nabla r\circ\nabla p=0$, $\nabla p\circ\nabla s=0$, and Eq. (1) becomes

$$\begin{cases}\varepsilon\left(\dfrac{\partial u}{\partial\tau}+\dfrac{\partial u}{\partial r}r_t+\dfrac{\partial u}{\partial s}s_t+\dfrac{\partial u}{\partial p}p_t\right)=\varepsilon^2\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{\partial^2 u}{\partial s^2}|\nabla s|^2+\dfrac{\partial^2 u}{\partial p^2}|\nabla p|^2+\right.\\ \qquad\left.\dfrac{\partial u}{\partial r}\nabla^2 r+\dfrac{\partial u}{\partial s}\nabla^2 s+\dfrac{\partial u}{\partial p}\nabla^2 p\right)+f(u,v),\\ \dfrac{\partial v}{\partial\tau}+\dfrac{\partial v}{\partial r}r_t+\dfrac{\partial v}{\partial s}s_t+\dfrac{\partial v}{\partial p}p_t=\varepsilon\tau\left(\dfrac{\partial^2 v}{\partial r^2}+\dfrac{\partial^2 v}{\partial s^2}|\nabla s|^2+\dfrac{\partial^2 v}{\partial p^2}|\nabla p|^2+\right.\\ \qquad\left.\dfrac{\partial v}{\partial r}\nabla^2 r+\dfrac{\partial v}{\partial s}\nabla^2 s+\dfrac{\partial v}{\partial p}\nabla^2 p\right)+g(u,v).\end{cases} \tag{3}$$

As variable u changes little in neighborhood of tangent plane π and change fast in normal direction. We observe the change in expanded normal direction $r=\varepsilon R$ and obtain the approximation of order $O(\varepsilon^0)$:

$$\frac{\partial^2 u}{\partial R^2}+(\varepsilon \nabla^2 r-r_t)\frac{\partial u}{\partial R}+f(u,v)=0, \qquad \frac{\partial v}{\partial R}r_t=\tau\frac{\partial^2 v}{\partial R^2}. \tag{4}$$

As variable v changes little in the neighborhood of wave front π we set $v=v_0$ as constant. Eq. (4) can be simplified as follows:

$$\frac{\partial^2 u}{\partial R^2}+(\varepsilon \nabla^2 r-r_t)\frac{\partial u}{\partial R}+f(u,v_0)=0. \tag{5}$$

As the range of variable R is $(-\infty,+\infty)$ and the variable u in the normal direction of π jump from $u_-(v_0)$ to $u_+(v_0)$ we obtain

$$\lim_{R\to-\infty}\frac{\partial u}{\partial R}=\lim_{R\to+\infty}\frac{\partial u}{\partial R}=0. \tag{6}$$

From the property of reaction diffusion equation the characteristic velocity c of π satisfies the following equation:

$$r_t=\varepsilon\nabla^2 r+c \tag{7}$$

and $c=\int_{u_-(v_0)}^{u_+(v_0)} f(u,v_0)\mathrm{d}u \Big\backslash \int_{-\infty}^{+\infty}\left(\frac{\partial u}{\partial R}\right)^2\mathrm{d}R.$

As r denotes the normal distance from a point in the neighborhood of π to wave front π the variable r_t denotes the normal velocity of π. Constant c denotes the plane characteristic velocity of π. The variable $\nabla^2 r$ has relationship with mean curvature of π. So Eq. (7) denote the normal velocity of π influenced by curvature factor. And we called it the characteristic equation of wave front of excitable media. The relation between normal velocity of π and curvature factor first obtained by Keener in Ref. [8] and by Ding Da-fu and Grindrod[10, 11] in three dimensional space, respectively. Both of their results and Eq. (7) describe the relation between normal velocity and curvature factor of wave front π. But they have essential difference in understanding of curvature and in form of equation. The most important is that Eq. (7) has clean form, clear meaning and can be deeply theoretically studied. We rewrite Eq. (7) as

$$\frac{\partial r}{\partial t}=\varepsilon\left(\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}\right)+c. \tag{8}$$

As variable r denotes the normal distance from wave front π to point (x,y,z). The set $\left\{(x,y,z): r(x,y,z,t)=0\right\}$ are wave front π. So the solution of Eq. (8) can obtain the wave front surface of π.

In the following we use following toroidal coordinates to analyze Eq. (8)

$$\begin{cases} x=(R+\rho\cos\psi)\cos\phi, \\ y=(R+\rho\cos\psi)\sin\phi, \\ z=\rho\sin\psi. \end{cases} \tag{9}$$

Put transformation (9) into Eq. (8)

$$\frac{\partial r}{\partial t}=\varepsilon\left[\frac{1}{R+\rho\cos^2\psi}\frac{\partial^2 r}{\partial\phi^2}+\frac{\partial^2 r}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2 r}{\partial\psi^2}+\left(\frac{1}{\rho}+\frac{\cos\psi}{R+\rho\cos\psi}\right)\frac{\partial r}{\partial\rho}-\frac{\partial r}{\partial\psi}\frac{\sin\psi}{\rho^2(R+\rho\cos\psi)}\right]+c. \tag{10}$$

For Eq. (10) we introduce $r=C_3t+A_3\phi+d(\rho,\psi)$ and obtain

$$\frac{\partial^2 d}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2 d}{\partial\psi^2}+\frac{\partial d}{\partial\rho}\left(\frac{1}{\rho}+\frac{\cos\psi}{R+\rho\cos\psi}\right)-\frac{\partial d}{\partial\psi}\frac{\sin\psi}{\rho(R+\rho\cos\psi)}=\frac{C_3-c}{\varepsilon}. \tag{11}$$

Suppose $R \gg \rho$ we observe Eq. (11) in $\rho = \varepsilon R\theta$. For approximate in order $O(\varepsilon^0)$

$$\frac{\partial^2 d}{\partial \theta^2} + \frac{1}{\theta}\frac{\partial d}{\partial \theta} + \frac{1}{\theta^2}\frac{\partial^2 d}{\partial \psi^2} = m. \tag{12}$$

From Eq. (12) we obtain $d(\theta, \psi) = B_3\psi + q_1\theta^2 + q_2\ln\theta$ $(4q_1 = m)$.

So the approximate solution of Eq. (10) in order $O(\varepsilon^0)$ is

$$r(\rho, \psi, \phi, t) = C_3 t + A_3\phi + B_3\psi + Q_3\rho^2 + Q_4\ln\rho + k_3 \qquad (0 < \rho \ll R). \tag{13}$$

So the wave front π contains the structure of the following:

$$Ct + A\phi + B\psi + Q\rho^2 + Q_0\ln\rho + K = 0. \tag{14}$$

Here A, B, C, Q, Q_0, K are constants.

In Fig. 1 R is a constant and ρ changes little. Put the special form of Eq. (14) $\phi = h\Psi$ into coordinate system (9) we obtain the twisted surface on the ring. Different h have different twisted waves and Fig. 1(a) is Möbius band, Figs. 1 (b)(c)(d) are twisted toroidal surface and stair shape.

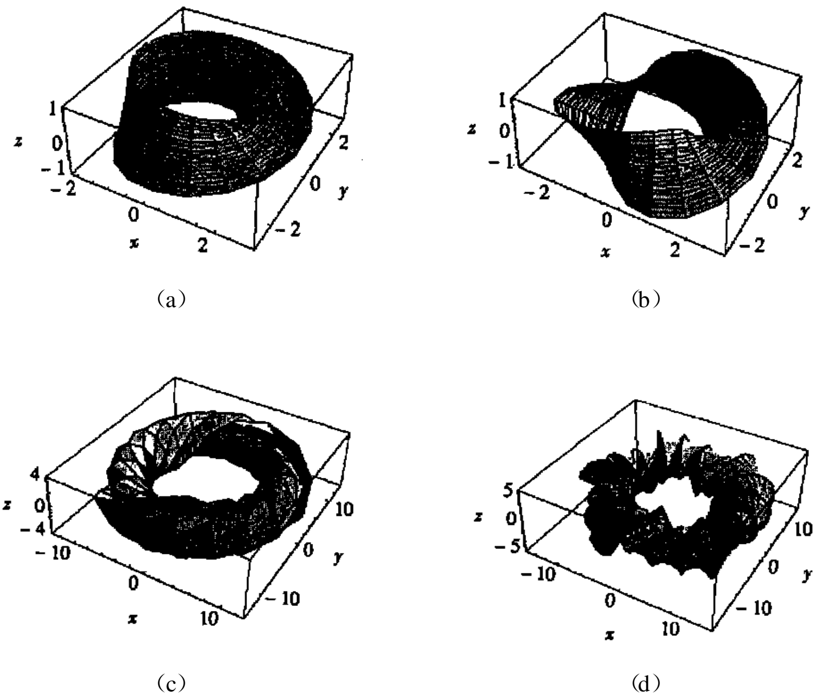


Fig. 1 The structure of wave front: (a) Möbius band; (b), (c), (d) Twisted toroidal surface and stair shape

In Fig. 1 we see waves front have structure of twisted toroidal surface. The odd twisted toroidal is Mobius band. The even twisted toroidal are twisted surface. When the twisted of wave front is serous wave, the wave front appears in stair state. The wave front surfaces in Figs. 1 (b)(c)(d) can be seen in experiments or numerical simulation[3,4] but the Möbius band needs testing.

The perturbation analysis before on wave front π can be applied on wave back surface. The same results are obtained for wave back. The intersection region of wave front and wave back can be abstracted as organization centers. The locus of organization center is

$$\begin{cases} \dfrac{\partial r_f}{\partial t} = \varepsilon\left(\dfrac{\partial^2 r_f}{\partial x^2} + \dfrac{\partial^2 r_f}{\partial y^2} + \dfrac{\partial^2 r_f}{\partial z^2}\right) + c_f, \\ \dfrac{\partial r_b}{\partial t} = \varepsilon\left(\dfrac{\partial^2 r_b}{\partial x^2} + \dfrac{\partial^2 r_b}{\partial y^2} + \dfrac{\partial^2 r_b}{\partial z^2}\right) + c_b. \end{cases} \tag{15}$$

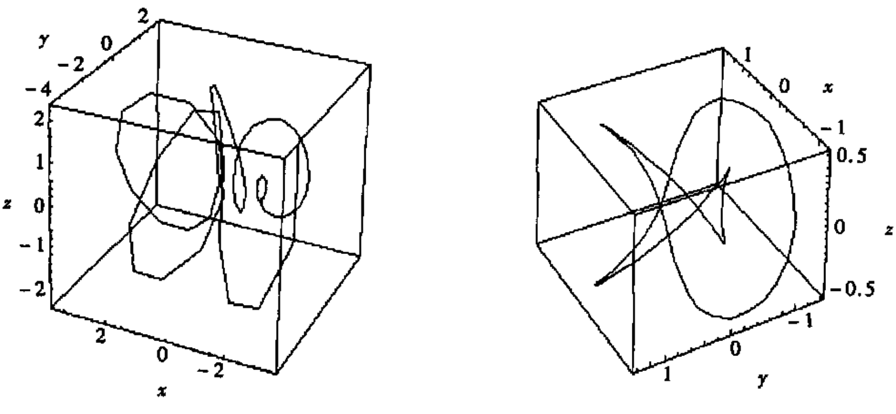
There are too many researches about organization center[5,8,9]. According to the author' s knowledge, this is the first characteristic equation satisfied by organization center. The null line of Eq. (15) denotes the wave front and wave back, respectively. The intersection of these two lines is organization center.

In toroidal coordinates the organization center contains the following structures:

$$\begin{cases} C_f t + A_f \phi + B_f \psi + Q_f \rho^2 + Q_{f0} \ln \rho + K_f = 0, \\ C_b t + A_b \phi + B_b \psi + Q_b \rho^2 + Q_{b0} \ln \rho + K_b = 0. \end{cases} \tag{16}$$

From the above characteristic equation we say the organization center rotates in spiral and moves up at the same time. These resemble the locus of wave tip. So we can understand organization center as compound of motion of wave tip and motion of moving up. When the time changes we can get the locus of motion.

In addition, the characteristic Eq. (15) has structure of (p, q) knotted on the toroidal surface. The simple Figs. 1(c)(d) knotted structure obtained in Refs. [8, 9] in numerical simulation or theoretic study. Here we obtain the general theoretic results first.



(a) (b)

Fig. 2 The locus of organization centers, (a) Rise spiral structure. (b) Knotted organization center[4,7]

2 Conclusions

In this paper, the perturbation method has been used to obtain the equation satisfied by wave front and organization centers of excitable media. The analysis here states that the wave front surface contain structure of scroll wave, trumpet shell and stair surface etc. The ring surface and their twisted form different structure. The even twisted toroidal are twisted surface. The odd twisted toroidal is Mobius band. When the twisted is serous wave the front appears in stair state.

These structures partly appear in the BZ experiments or numerical simulation. The structure of Möbius band needs testing.

When we consider the organization center as intersection of wave front and wave back we obtain the characteristic equation of organization center. Their locus can be understood as compound of motion of tip and motion of verticality. The special case is (p, q) type twisted on the ring surface.

Möbius band as wave front is an interesting result. If we cut the Möbius band from its center we obtain the twisted ring surface. When cut this ring surface twice we get two twisted, linked ring surfaces. If we consider this two ring surface as wave surface we obtain two twisted and linked organization centers. The simulation results[9] reveal the organization centers have structure of tangle, linked states and the link is in ring. Obviously the conclusion here is deeper and more definite.

The characteristic equation of wave front and organization center here provides plenty of wave structure. Large numbers of simulation and experiment results are parts of our theoretic results. There is too much work to do to study characteristic equation of wave front here.

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