SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 14, No. 4, pp. 1808–1844

Averaging, Folded Singularities, and Torus Canards: Explaining Transitions between Bursting and Spiking in a Coupled Neuron Model*

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Abstract. In this work, we identify generic bifurcation scenarios corresponding to transitions between bursting and tonic spiking solutions in a model for a coupled pair of burst-capable neurons, and we elucidate the central role of folded singularities in these scenarios. The folded singularities in our work arise in the context of fast-slow averaging, and hence our results link with the study of torus canards, a recently identified class of ordinary differential equation (ODE) solutions featuring oscillatory excursions along repelling structures in phase space [J. Burke et al., *J. Math. Neurosci.*, 2 (2012), pp. 1–30]; in particular, our work extends this study to systems featuring two slow variables and symmetry and goes significantly beyond the analysis of activity transitions presented by Best et al. [*SIAM J. Appl. Dyn. Syst.*, 4 (2005), pp. 1107–1139].

Key words. geometric singular perturbation theory, multiple time scales, averaging, folded singularities, torus canards, bursting, neuronal dynamics

AMS subject classifications. 37N25, 34E17, 92B25, 37M05, 37G35

DOI. 10.1137/140981770

1. Introduction.

Bursting behavior in neurons has been the focus of significant theoretical attention due to both its mathematical complexity and its central role in driving repetitive actions such as respiration and hormone release (see, e.g., [9]). A wide variety of forms of bursting that arise in two time scale single-neuron models are well understood based on fast-slow decomposition, identification of fast subsystem bifurcation structures, and averaging [27, 20], and these methods can also be used to explain transitions between quiescence, certain forms of bursting, and tonic spiking in single neurons (see, e.g., [36, 31, 17]). Transitions between such activity patterns in neuronal network models, however, are much less well understood. In this work, we identify generic bifurcation scenarios corresponding to transitions between bursting and tonic spiking solutions in a model for a coupled pair of burst-capable neurons, and we elucidate the central role of *folded singularities* [13, 38, 34] and *canards* in these scenarios. The folded singularities in our work arise in the context of fast-slow averaging, and hence our results link with the study of *torus canards*, a recently identified class of ordinary differential equation (ODE) solutions featuring oscillatory excursions along repelling structures in phase space [21, 2]; in particular, our work extends this study to systems featuring two slow variables

*Received by the editors August 11, 2014; accepted for publication (in revised form) July 29, 2015; published electronically October 20, 2015.
http://www.siam.org/journals/siads/14-4/98177.html

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